ON THE REAL MILNOR FIBRE OF SOME MAPS FROM $\mathbb{R}^n$ TO $\mathbb{R}^2$

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Abstract. We consider a real analytic map-germ $(f,g) : (\mathbb{R}^n,0) \to (\mathbb{R}^2,0)$. Under some conditions, we establish degree formulas for the following quantities :

$$\chi(\{f=\alpha\} \cap \{g=\delta\} \cap B_\varepsilon^n),$$

$$\chi(\{f=\alpha\} \cap \{g \geq \delta\} \cap B_\varepsilon^n) - \chi(\{f=\alpha\} \cap \{g \leq \delta\} \cap B_\varepsilon^n),$$

where (α,δ) is a regular value of (f,g) and $0 < |(\alpha,\delta)| \ll \varepsilon \ll 1$.

1. Introduction

Let $F=(f_1,\ldots,f_k) : (\mathbb{K}^n,0) \to (\mathbb{K}^k,0)$, with $0<k<n$ and $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, be an analytic map-germ such that 0 is an isolated singularity of $F^{-1}(0)$, i.e rank$[DF(x)] = k$ at every $x \in F^{-1}(0) \setminus \{0\}$ close to the origin. Let $g : (\mathbb{K}^n,0) \to (\mathbb{K},0)$ be a function-germ. We are interested in studying topological invariants associated with the mappings F and (F,g).

Let $B_\varepsilon \subset \mathbb{K}^n$ be a closed ball centered at the origin of radius ε. For any regular value $\delta \in \mathbb{K}^k$ close to the origin, the Milnor fibre of F associated with δ is the set $F^{-1}(\delta) \cap B_\varepsilon$, where $0<|\delta| \ll \varepsilon \ll 1$. We will denote it by $W^\varepsilon_{F-\delta}$.

In the complex case, the topology of the Milnor fibre is well known. Milnor [Mi] (in the case $k=1$) and Hamm [Ha] (in the case $k>1$) proved that $W^\varepsilon_{F-\delta}$ has the homotopy type of a bouquet of $\mu(F)$ spheres of dimension $n-k$. This number $\mu(F)$ is called the Milnor number of F and the Euler-Poincaré characteristic of the Milnor fibre is given by $\chi(W^\varepsilon_{F-\delta}) = 1+(-1)^{n-k}\mu(F)$. Minor also showed that, when $k=1$, $\mu(F)$ is equal to the topological degree of the mapping $\frac{\nabla F}{|\nabla F|} : \partial B_\varepsilon \to S_1^{2n-1}$. This gives the following algebraic characterization of the Milnor number :

$$\mu(F) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,0}}{\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)},$$

where $\mathcal{O}_{\mathbb{C}^n,0}$ is the algebra of holomorphic function-germs at the origin.

This last formula was extended to the case $k>1$ by Lê [Le] and Greuel [Gr], who obtained the following formula :

$$\mu(F') + \mu(F) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/I,$$

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where $F' = (f_1, \dots, f_{k-1})$ and I is the ideal generated by $f_1, \dots, f_{k-1}$ and the minors $\frac{\partial(f_1,\dots,f_k)}{\partial(x_{i_1},\dots,x_{i_k})}$. Hence, proceeding by induction, one can get a formula for $\mu(F)$ in terms of dimensions of appropriate local algebras.

In the real case, the topology of the Milnor fibre depends on δ and can not be as well described as in the complex case. Nevertheless, there exist formulas similar to the ones mentionned above relating its Euler-Poincaré characteristic to topological degrees of mapping defined in terms of F. For instance, if $k = 1$, the Khimshiashvili formula [Kh] states that :

$$\chi(W^{\varepsilon}_{F-\delta}) = 1 - \mathrm{sign}(-\delta)^n \cdot \deg_0 \nabla F,$$

where $\deg_0 \nabla F$ is the topological degree of the mapping $\frac{\nabla F}{|\nabla F|} : \partial B_\varepsilon \to S_1^{n-1}$.

We proved in [Du2] that if (δ, α) is a regular value of (F, g) with $0 < |\alpha| \ll |\delta| \ll \varepsilon \ll 1$ then :

$$\begin{aligned} &\chi(W^{\varepsilon}_{F-\delta} \cap \{g \geq \alpha\}) - \chi(W^{\varepsilon}_{F-\delta} \cap \{g \leq \alpha\}) \equiv \\ &\qquad \chi(W^{\varepsilon}_{(F-\delta, g-\alpha)}) + \chi(W^{\varepsilon}_{F-\delta}) \equiv \dim_{\mathbb{R}} \mathcal{O}_{\mathbb{R}^n,0}/I \bmod (2), \end{aligned}$$

where $\mathcal{O}_{\mathbb{R}^n,0}$ is the ring of real analytic function-germs at the origin and I is the ideal generated by $f_1, \dots, f_k$ and the minors $\frac{\partial(g,f_1,\dots,f_k)}{\partial(x_{i_1},\dots,x_{i_{k+1}})}$. This formula generalized the case $g = x_1^2 + \dots + x_n^2$ already shown by Duzinski et al. [DLNS]. This a mod 2 formula and one may ask if, as in the complex case, it is possible to express the following quantities :

$$\chi(W^{\varepsilon}_{F-\delta} \cap \{g \geq \alpha\}) - \chi(W^{\varepsilon}_{F-\delta} \cap \{g \leq \alpha\}),$$

and

$$\chi(W^{\varepsilon}_{F-\delta}),$$

in terms of topological degrees of mappings defined in terms of F and g.

When $k = n-1$ and $g = x_1^2 + \dots + x_n^2$, Aoki et al. ([AFN1],[AFS]) proved that : $\chi(W^{\varepsilon}_{F-\delta}) = \deg_0 H$ and $2 \times \deg_0 H$ is the number of half-branches of $F^{-1}(0)$, where $H = \left(\frac{\partial(g,f_1,\dots,f_{n-1})}{\partial(x_1,\dots,x_n)}, f_1, \dots, f_{n-1}\right)$. They extended this result to the case $g = x_n$ in [AFN2] and Szafraniec extended it to any g in [Sz1].

When $k = 1$ and $g = x_1$, Fukui showed in [Fu] that :

$$\chi(W^{\varepsilon}_{F-\delta} \cap \{x_1 \geq 0\}) - \chi(W^{\varepsilon}_{F-\delta} \cap \{x_1 \leq 0\}) = -\mathrm{sign}(-\delta)^n \cdot \deg_0 H,$$

where $H = (F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$.

In [Du1], we proved that when $n = 2, 4$ or 8 and $k = 1$, it is possible to construct a mapping $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that :

$$\chi(W^{\varepsilon}_{F-\delta} \cap \{g \geq \alpha\}) - \chi(W^{\varepsilon}_{F-\delta} \cap \{g \leq \alpha\}) = -\deg_0 H.$$

This last result was generalized by Fukui and Khovanskii [FK]. In that paper, the authors consider an analytic function-germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ that satisfies the following condition (P) : there exits C^∞-vector fields $v_2, \dots, v_n$ which span the tangent space at x to $g^{-1}(g(x))$, whenever x is a regular point of g, and $\nabla g, v_2, \dots, v_n$ agree with the orientation of $\mathbb{R}^n$. They define a mapping $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ by $H = (F, v_2 F, \dots, v_n F)$ and they prove that if 0 is isolated in $H^{-1}(0)$, if the set of critical points of g does not intersect $W^\varepsilon_{F-\delta}$ and if $(\delta, 0)$ is a regular value of (F, g) then :

$$\chi(W^\varepsilon_{F-\delta} \cap \{g \geq 0\}) - \chi(W^\varepsilon_{F-\delta} \cap \{g \leq 0\}) = \mathrm{sign}(-\delta)^n \cdot \deg_0 H.$$

In this paper, we continue this work of computing Euler-Poincaré characteristics of real Milnor fibres, especially for mappings with two components. In Section 2, we give generalizations of Khimshiashvili's formula. We consider an analytic function-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, with an isolated critical point at 0, that satisfies the condition (P) described above. Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an other function-germ. We define a mapping $k(f, g) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ in terms of f and g and we assume that it has an isolated zero at the origin. We prove (Theorem 2.1) that :

$$\text{if } n \text{ is even } : \chi(W^\varepsilon_{(f,g-\delta)}) = 1 - \deg_0 \nabla f + \mathrm{sign}(\delta) \cdot \deg_0 k(f, g),$$
$$\text{if } n \text{ is odd } : \chi(W^\varepsilon_{(f,g-\delta)}) = 1 - \deg_0 k(f, g).$$

We also show that if n is even :

$$\chi(W^\varepsilon_{f-\alpha} \cap \{g \geq \delta\}) - \chi(W^\varepsilon_{f-\alpha} \cap \{g \leq \delta\}) = \deg_0 k(f, g),$$

where (α, δ) is an appropriate regular value of (f, g). Then we assume that g has an isolated critical point at the origin as well and we define an other mapping $l(f, g)$. If it has an isolated zero at the origin, then we have (Theorem 2.9) :

$$\text{if } n \text{ is even } : \chi(W^\varepsilon_{(f-\delta,g)}) = 1 - \deg_0 \nabla g - \mathrm{sign}(\delta) \cdot \deg_0 l(f, g),$$
$$\text{if } n \text{ is odd } : \chi(W^\varepsilon_{f-\delta} \cap \{g \geq 0\}) - \chi(W^\varepsilon_{f-\delta} \cap \{g \leq 0\}) =$$
$$\deg_0 \nabla g + \mathrm{sign}(\delta) \cdot \deg_0 l(f, g),$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

In Section 3, we give a generalization of the formula of Fukui mentionned above. We work in $\mathbb{R}^{1+n}$ equipped with the coordinate system $(x_0, x_1, \ldots, x_n)$ and we consider a function-germ $F : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ with an isolated critical point at 0. We assume that F satisfies the following condition (Q) : there exists C^∞ vector fields $V_2, \ldots, V_n$ on $\mathbb{R}^{1+n}$ such that $V_2(p), \ldots, V_n(p)$ span the tangent space at p to $F^{-1}(F(p)) \cap x_0^{-1}(x_0(p))$ whenever p is a regular point of (F, x_0) and such that $(e_0, \nabla F(p), V_2(p), \ldots, V_n(p))$ agrees with the orientation of $\mathbb{R}^{1+n}$. Here e_0 is the vector $(1, 0, \ldots, 0)$. Let $G : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be an other function-germ. We define three mappings $H(F, G)$, $I(F, G)$ and $J(F, G) : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}^{1+n}, 0)$. We prove that if 0 is isolated in $J(F, G)^{-1}(0)$ and $(0, \delta, 0)$ is a regular value of (F, G, x_0) then (Theorem 3.1) :

$$\deg_0 H(F, G) = \mathrm{sign}(-\delta)^n \cdot \Big[\chi\left(W^\varepsilon_{(F,G-\delta)} \cap \{x_0 \geq 0\}\right)$$
$$-\chi\left(W^\varepsilon_{(F,G-\delta)} \cap \{x_0 \leq 0\}\right)\Big],$$

$$\deg_0 J(F, G) = \mathrm{sign}(-\delta)^n \cdot \left[\chi\left(W^\varepsilon_{(F,G-\delta)}\right) - \chi\left(W^\varepsilon_{(G-\delta,x_0)}\right)\right],$$

where $0 < |\delta| \ll \varepsilon \ll 1$. Then we apply these formulas to the case where F and G are one-parameter deformations of two function-germs f and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Denoting by f_t and g_t the deformations given by $f_t(x) = F(t,x)$ and $g_t(x) = G(t,x)$ and applying a deformation argument as Fukui did in [Fu], we obtain degree formulas for $\chi(W^\varepsilon_{(f_t,g_t)})$ and

$$\chi(W^\varepsilon_{f_t} \cap \{g_t \geq 0\}) - \chi(W^\varepsilon_{f_t} \cap \{g_t \leq 0\}),$$

where $0 < |t| \ll \varepsilon \ll 1$ (Corollary 3.15). When the deformations of f and g are of the form :

$$F(t,x) = f(x) - \gamma_1(t) \text{ and } G(t,x) = g(x) - \gamma_2(t),$$

where $\gamma = (\gamma_1, \gamma_2) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is an analytic arc such that $\gamma(t) \neq 0$ if $t \neq 0$, $\gamma_1'(t) \neq 0$ if $t \neq 0$, and the image of γ consists of regular values of (f,g) (except the origin in $\mathbb{R}^2$ of course), we get formulas for $\chi(W^\varepsilon_{(f-\gamma_1(t),g-\gamma_2(t))})$ and

$$\chi(W^\varepsilon_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^\varepsilon_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}),$$

where $0 < |t| \ll \varepsilon \ll 1$ (Corollary 3.17).

In Section 4, we present different cases where we can apply the results of the previous sections. There are two cases : when $n = 2, 4$ or 8 and when $\frac{\partial f}{\partial x_1} \geq 0$ and $\frac{\partial F}{\partial x_1} \geq 0$. In this last situation, we explain how the results concerning the one-parameter deformations can be simplified with the aid of Theorem 2.1 (see Corollaries 4.1 and 4.2).

We will use the following notations. Some of them have already appeared in this introduction :

(1) if $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a mapping with an isolated zero at the origin then $\deg_0 H$ is the topological degree of $\frac{H}{|H|} : S^{n-1}_\varepsilon \to S^{n-1}_1$ where S^{n-1}_ε is the sphere of radius ε centered at the origin,
(2) if $F : \mathbb{R}^n \to \mathbb{R}^k$ is a mapping then W^ε_F denotes the set $F^{-1}(0) \cap B^n_\varepsilon$, where B^n_ε is the ball of radius ε centered at the origin and ∂W^ε_F is $F^{-1}(0) \cap S^{n-1}_\varepsilon$,
(3) if $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function then f_{x_i} denotes the partial derivative $\frac{\partial f}{\partial x_i}$ and ∇f is the gradient of f,
(4) if $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$, $0 < k \leq n$, is a smooth mapping then $DF(x)$ is its Jacobian matrix at x and $\frac{\partial(F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_k})}$ is the determinant of the following $k \times k$ minors of $DF(x)$:

$$\begin{pmatrix} F_{1x_{i_1}} & \cdots & F_{1x_{i_k}} \\ \vdots & \ddots & \vdots \\ F_{kx_{i_1}} & \cdots & F_{kx_{i_k}} \end{pmatrix}.$$

2. Generalizations of Khimshiashvili's formula

In this section, we prove formulas similar to Khimshiashvili's one for the fibre of a function on a hypersurface with an isolated singularity. We need to put the same conditon (P) as Fukui and Khovanskii's condition (P) [FK], either on the function or on the function defining the hypersurface.

Let $(x_1, \ldots, x_n)$ be a coordinate system in $\mathbb{R}^n$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. We assume that f satisfies the following condition (P) : there exists C^∞ vector fields $v_2, \ldots, v_n$ on $\mathbb{R}^n$ such that $v_2(x), \ldots, v_n(x)$ span the tangent space at x to $f^{-1}(f(x))$, whenever x is a regular point of f, and such that the orientation of $(\nabla f(x), v_2(x), \ldots, v_n(x))$ agrees with the orientation of $\mathbb{R}^n$.

Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an other analytic function-germ. We define a mapping $k(f,g) : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ in the following way :

$$k(f,g) = (f, v_2 g, \ldots, v_n g).$$

We will prove the following theorem :

Theorem 2.1. *If* 0 *is an isolated critical point of* f *and is isolated in* $k(f,g)^{-1}(0)$*, then we have :*

$$\text{if } n \text{ is even } : \chi(W^{\varepsilon}_{(f,g-\delta)}) = 1 - \deg_0 \nabla f + \operatorname{sign}(\delta) \cdot \deg_0 k(f,g),$$
$$\text{if } n \text{ is odd } : \chi(W^{\varepsilon}_{(f,g-\delta)}) = 1 - \deg_0 k(f,g),$$

where $0 < |\delta| \ll \varepsilon \ll 1$*. Furthermore, if* n *is even, we also have :*

$$\chi(W^{\varepsilon}_{f-\alpha} \cap \{g \geq \delta\}) - \chi(W^{\varepsilon}_{f-\alpha} \cap \{g \leq \delta\}) = \deg_0 k(f,g),$$

where $0 \leq |\delta| \ll |\alpha| \ll \varepsilon \ll 1$ *and* (α, δ) *is a regular value of* (f,g)*.*

To establish this theorem, we need a series of lemmas. From now on, we will assume that the hypothesis of Theorem 2.1 are fullfilled. For all $(i,j) \in \{1, \ldots, n\}^2$, we will set $m_{ij} = \frac{\partial(g,f)}{\partial(x_i,x_j)}$.

Lemma 2.2. *For* $\delta \neq 0$ *sufficiently small,* $(0, \delta)$ *is a regular value of* (f,g)*.*

Proof. Since f has an isolated critical point, $f^{-1}(0) \setminus \{0\}$ is smooth (or empty). By the Curve Selection Lemma, the critical points of $g_{|f^{-1}(0)\setminus\{0\}}$ lie in $g^{-1}(0)$. □

Lemma 2.3. *Let* p *be a regular point of* f*. The function* $g_{|f^{-1}(f(p))}$ *has a critical point at* p *if and only if* $v_i g(p) = 0$ *for all* $i \in \{2, \ldots, n\}$*.*

Proof. If p is a regular point of f then $v_2(p), \ldots, v_n(p)$ span the tangent space at $f^{-1}(f(p))$. Therefore $g_{|f^{-1}(f(p))}$ has a critical point at p if and only if $\langle v_i(p), \nabla g(p) \rangle = 0$ for all $i \in \{2, \ldots, n\}$. □

Lemma 2.4. *The origin is an isolated singularity of* $f^{-1}(0) \cap g^{-1}(0)$ *if and only if* 0 *is isolated in* $k(f,g)^{-1}(0)$*.*

Proof. A point p, distinct from the origin, is in $k(f,g)^{-1}(0)$ if and only if $g_{|f^{-1}(0)\setminus\{0\}}$ has a critical point at p. But, as noticed above, such a point lies in $g^{-1}(0)$. □

Lemma 2.5. *Let $\alpha \neq 0$ be a small regular value of f. Let p be a point in $f^{-1}(\alpha)$. The function $g_{|f^{-1}(\alpha)}$ has a non-degenerate critical point at p if and only if $k(f,g)(p) = (\alpha, 0, \ldots, 0)$ and $\det Dk(f,g)(p) \neq 0$. Furthermore if $\lambda(p)$ is the Morse index of $g_{|f^{-1}(\alpha)}$ at p then one has :*

$$(-1)^{\lambda(p)} = \text{sign}\left[\det Dk(f,g)(p)\right].$$

Proof. Since α is a regular value of f, there exists j such that $f_{x_j}(p) \neq 0$. Assume that $j = 1$. From [Sz2,p349-350], p is a non-degenerate critical point of $g_{|f^{-1}(\alpha)}$ if and only if :

$$\det \begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}_{2 \leq i \leq n} \neq 0.$$

Furthermore, we have :

$$(-1)^{\lambda(p)} = (-1)^{n-1} \cdot \text{sign}(f_{x_1}(p))^n \cdot \det \begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}.$$

We have to relate $\det \begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}$ to $\det \begin{bmatrix} \nabla f(p) \\ \nabla v_i g(p) \end{bmatrix}$. For $i \in \{2, \ldots, n\}$, let $u_i(p)$ be the vector $(f_{x_i}(p), 0, \ldots, 0, -f_{x_1}(p), 0, \ldots, 0)$, where $-f_{x_1}(p)$ is the i-th coordinate. Then $(u_2(p), \ldots, u_n(p))$ is a basis of $T_p f^{-1}(\alpha)$ and it is not difficult to see that :

$$\det(\nabla f(p), u_2(p), \ldots, u_n(p)) = (-1)^{n-1} \cdot f_{x_1}(p)^{n-2} \cdot (\sum_{i=1}^{n} f_{x_i}^2(p)).$$

Hence there exists a $(n-1) \times (n-1)$ matrix $B(p)$ such that :

$$\begin{pmatrix} \nabla f(p) \\ u_i(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B(p) \end{pmatrix} \cdot \begin{pmatrix} \nabla f(p) \\ v_i(p) \end{pmatrix},$$

with $\text{sign}[\det B(p)] = (-1)^{n-1} \text{sign}[f_{x_1}(p)^{n-2}]$. Hence, for $i \in \{2, \ldots, n\}$:

$$u_i(p) = \sum_{j=2}^{n} B_{ij}(p) \cdot v_j(p),$$

and :

$$m_{1i}(p) = u_i g(p) = \sum_{j=2}^{n} B_{ij}(p) \cdot v_j g(p).$$

Since $v_j g(p) = 0$, we have :

$$\nabla m_{1i}(p) = \sum_{j=2}^{n} B_{ij}(p) \cdot \nabla v_j g(p),$$

and :

$$\begin{pmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B(p) \end{pmatrix} \cdot \begin{pmatrix} \nabla f(p) \\ \nabla v_i g(p) \end{pmatrix}.$$

With this equality, it is easy to conclude. $\square$

To prove Theorem 2.1, we will use Morse theory for manifolds with corners. The reader may refer to [Du4,Section 2] for a brief description of this theory. The following lemma deals with the critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$.

Lemma 2.6. *For all α and ε such that $0 < |\alpha| \ll \varepsilon \ll 1$, one has :*

- *at all correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ with $g > 0$, $\nabla g_{|f^{-1}(\alpha)}$ points outwards,*
- *at all correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ with $g < 0$, $\nabla g_{|f^{-1}(\alpha)}$ points inwards,*
- *there are no correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ in $g^{-1}(0)$.*

Proof. The proof is the same as [Du1], Lemma 4.1. □

Lemma 2.7. *We can choose α small enough and we can perturbe g into $\tilde{g}$ in such a way that $\tilde{g}_{|W^{\varepsilon}_{f-\alpha}}$ has only Morse critical points.*

Proof. Let $(x;t) = (x_1, \ldots, x_n; t_1, \ldots, t_n)$ be a coordinate system of $\mathbb{R}^{2n}$ and let $\bar{g}(x,t) = g(x) + t_1 x_1 + \cdots t_n x_n$. For $(i,j) \in \{1, \ldots, n\}^2$, we define $M_{ij}(x,t)$ by $M_{ij}(x,t) = \frac{\partial(f,\bar{g})}{\partial(x_i,x_j)}(x,t)$. Notice that :

$$M_{ij}(x,t) = m_{ij}(x,t) + f_{x_i}(x) t_j - t_i f_{x_j}(x).$$

Let Γ be defined by :

$$\Gamma = \left\{ (x,t) \in \mathbb{R}^{2n} \mid M_{ij}(x,t) = 0 \text{ for } (i,j) \in \{1, \ldots, n\}^2 \right\}.$$

At a point p, if f does not vanish then there exists $i \in \{1, \ldots, n\}$ such that $f_{x_i}(p) \neq 0$. This implies that $\Gamma \setminus \{f = 0\}$ is a smooth manifold (or empty) of dimension $n+1$. Actually if p belongs to $\Gamma \setminus \{f = 0\}$, then one can assume that $f_{x_1}(p) \neq 0$. In this case, around p, Γ is defined by the vanishing of $M_{12}, \ldots, M_{1n}$ and the gradient vector fields of these functions are linearly independent. Let π be the following mapping :

$$\begin{array}{cccc} \pi & : & \Gamma \setminus \{f = 0\} & \rightarrow & \mathbb{R}^{1+n} \\ & & (x,t) & \mapsto & (f(x), t). \end{array}$$

By the Bertini-Sard theorem, we can choose (α, s) close to 0 in $\mathbb{R}^{1+n}$ such that π is regular at each point in $\pi^{-1}(\alpha, s)$ close to the origin. If we denote by $\tilde{g}$ the function defined by $\tilde{g}(x) = \bar{g}(x,s)$, this means that $\tilde{g}_{|f^{-1}(\alpha)}$ admits only Morse critical points in a neighborhood of the origin. □

Proof of Theorem 2.1. Let $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance function to the origin. Let $\varepsilon > 0$ be sufficiently small so that $g_{|f^{-1}(0) \setminus \{0\}}$ has no critical point in $f^{-1}(0) \setminus \{0\} \cap \{\omega < \varepsilon\}$. Let δ be such that $0 < |\delta| \ll \varepsilon \ll 1$. We want to express $\chi(W^{\varepsilon}_{(f,g-\delta)})$ in terms of $\deg_0 k(f,g)$. Let α be a regular value of f such that $0 < |\alpha| \ll |\delta|$ and the following properties are satisfied :

(1) $W^{\varepsilon}_{(f-\alpha,g-\delta)}$ is diffeomorphic to $W^{\varepsilon}_{(f,g-\delta)}$,
(2) the critical points of $g_{|f^{-1}(\alpha) \cap \{\omega < \varepsilon\}}$ lie in $\{|g| < \delta\} \cap \{\omega < \frac{\varepsilon}{2}\}$.

Hence the critical points of $g_{|\partial W^\varepsilon_{f-\alpha}}$ are correct. Furthermore by the previous lemmas, we can assume that $g_{|f^{-1}(\alpha)\cap\{\omega<\varepsilon\}}$ has only Morse critical points, that at the correct critical points of $g_{|W^\varepsilon_{f-\alpha}}$ lying in $\{g>0\}$ (resp. $\{g<0\}$), $\nabla g_{|W^\varepsilon_{f-\alpha}}$ points outwards (resp. inwards) and that there are no correct critical points of $g_{|W^\varepsilon_{f-\alpha}}$ in $g^{-1}(0)$.

We assume that $\delta>0$ and we apply Morse theory for manifolds with boundary to obtain :

$$\chi(W^\varepsilon_{f-\alpha}\cap\{g\geq-\delta\},W^\varepsilon_{(f-\alpha,g+\delta)})=\sum_i(-1)^{\lambda(p_i)},$$

where $\{p_i\}$ is the set of critical points of $g_{|f^{-1}(\alpha)\cap\{\omega<\varepsilon\}}$, and :

$$\chi(W^\varepsilon_{f-\alpha}\cap\{g\leq-\delta\},W^\varepsilon_{(f-\alpha,g+\delta)})=0.$$

Summing these equalities and using the Mayer-Vietoris sequence gives :

$$\chi(W^\varepsilon_{f-\alpha})-\chi(W^\varepsilon_{(f-\alpha,g+\delta)})=\sum_i(-1)^{\lambda(p_i)}.$$

By Lemma 2.5, $\sum_i(-1)^{\lambda(p_i)}$ is equal to $\deg_0 k(f,g)$. By Khimshiashvili's formula, $\chi(W^\varepsilon_{f-\alpha})=1-\text{sign}(-\alpha)^n\cdot\deg_0\nabla f$. Now by Proposition 1.1 in [FK], we know that $\deg_0\nabla f=0$ if n is odd. This gives the result for the fibre $W^\varepsilon_{(f,g-\delta)}$ with $\delta<0$. The formula for the fibre $W^\varepsilon_{(f,g-\delta)}$ with $\delta>0$ is obtained replacing g with $-g$. It remains to prove the third formula. Let δ be such that (α,δ) is a regular value of (f,g) and $0\leq|\delta|\ll|\alpha|\ll\varepsilon$. Since n is even, we have :

$$\chi(W^\varepsilon_{f-\alpha}\cap\{g\geq\delta\})-\chi(W^\varepsilon_{(f-\alpha,g-\delta)})=\sum_{i|g(p_i)>\delta}(-1)^{\lambda(p_i)},$$

$$\chi(W^\varepsilon_{f-\alpha}\cap\{g\leq\delta\})-\chi(W^\varepsilon_{(f-\alpha,g-\delta)})=-\sum_{i|g(p_i)<\delta}(-1)^{\lambda(p_i)}.$$

Making the difference and using Lemma 2.5 leads to the result. □

Corollary 2.8. *If* 0 *is an isolated critical point of* f *and is isolated in* $k(f,g)^{-1}(0)$*, then one has :*

if n *is odd* : $\chi(\partial W^\varepsilon_{(f,g)})=2-2\deg_0 k(f,g)$,

if n *is even* : $\chi(\partial W^\varepsilon_f\cap\{g\geq0\})-\chi(\partial W^\varepsilon_f\cap\{g\leq0\})=\ 2\deg_0 k(f,g)$.

Proof. The first point is easy. For the second point, see [Du1], Theorem 5.2. □

Now let us suppose that g also has an isolated critical point at the origin and consider the mapping $l(f,g):(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ defined by :

$$l(f,g)=(g,v_2g,\ldots,v_ng).$$

In [FK], Theorem 4.1, Fukui and Khovanskii prove that if 0 is isolated in $l(f,g)^{-1}(0)$ and if the set of critical points of f does not intersect $W^\varepsilon_{g-\delta}$ then:

$$\deg_0 l(f,g)=-\text{sign}(-\delta)^n\cdot\left\{\chi(W^\varepsilon_{g-\delta}\cap\{f\geq0\})-\chi(W^\varepsilon_{g-\delta}\cap\{f\leq0\})\right\}.$$

In our situation the second condition is fullfilled because f has an isolated critical point. We want to give an other interpretation to this degree. We will prove the following theorem.

Theorem 2.9. *If f and g have an isolated critical point at the origin and 0 is isolated in $l(f,g)^{-1}(0)$ then :*

$$\begin{aligned} &\text{if } n \text{ is even } : \chi(W^{\varepsilon}_{(f-\delta,g)}) = 1 - \deg_0 \nabla g - \mathrm{sign}(\delta) \cdot \deg_0 l(f,g), \\ &\text{if } n \text{ is odd } : \chi(W^{\varepsilon}_{f-\delta} \cap \{g \geq 0\}) - \chi(W^{\varepsilon}_{f-\delta} \cap \{g \leq 0\}) = \\ &\qquad\qquad\qquad\qquad\qquad \deg_0 \nabla g + \mathrm{sign}(\delta) \cdot \deg_0 l(f,g), \end{aligned}$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

The proof of this theorem goes nearly like the proof of Theorem 2.1. We need some lemmas.

Lemma 2.10. *For $\delta \neq 0$ sufficiently small, $(\delta, 0)$ is a regular value of (f,g).*

Lemma 2.11. *Let p be a regular point of g. The function $f_{|g^{-1}(g(p))}$ has a critical point at p if and only if $v_i g(p) = 0$ for all $i \in \{2, \ldots, n\}$.*

Proof. The function $f_{|g^{-1}(g(p))}$ has a critical point at p if and only if $\nabla f(p)$ and $\nabla g(p)$ are colinear. Since these two vectors are non zero, this is equivalent to the fact that $g_{|f^{-1}(f(p))}$ has a critical point at p. It is enough to use Lemma 2.3. □

Lemma 2.12. *The origin is an isolated singularity of $f^{-1}(0) \cap g^{-1}(0)$ if and only if 0 is isolated in $l(f,g)^{-1}(0)$.*

Lemma 2.13. *Let $\alpha \neq 0$ be a small regular value of g. Let p be a point in $g^{-1}(\alpha)$. The function $f_{|g^{-1}(\alpha)}$ has a non-degenerate critical point at p if and only if $l(f,g)(p) = (\alpha, 0, \ldots, 0)$ and $\det Dl(f,g)(p) \neq 0$. Furthermore if $\lambda(p)$ is the Morse index of $f_{|g^{-1}(\alpha)}$ at p and if $\mu(p)$ is the real number such that $\nabla f(p) = \mu(p) \cdot \nabla g(p)$ then we have :*

$$(-1)^{\lambda(p)} = (-1)^{n-1} \cdot \mathrm{sign}\left[\mu(p)^n \cdot \det Dl(f,g)(p)\right].$$

Proof. Since α is a regular value of g, there exists j such that $g_{x_j}(p) \neq 0$. Assume that $j = 1$. From [Sz2,p349-350], p is a non-degenerate critical point of $f_{|g^{-1}(\alpha)}$ if and only if :

$$\det \begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix}_{2 \leq i \leq n} \neq 0.$$

Furthermore, we have :

$$(-1)^{\lambda(p)} = (-1)^{n-1} \cdot \mathrm{sign}\left(g_{x_1}(p)^n \cdot \det \begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix} \right).$$

Since $g_{x_1}(p) \neq 0$, $f_{x_1}(p)$ does not vanish for otherwise $\mu(p)$ and $\nabla f(p)$ would vanish as well. Then the computations done in Lemma 2.5 show that :

$$\mathrm{sign}\left(\det \begin{bmatrix} \nabla g(p) \\ -\nabla v_i g(p) \end{bmatrix} \right) = \mathrm{sign}\left(f_{x_1}(p)^{n-2} \cdot \det \begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix} \right),$$

and it is easy to conclude. $\square$

The following lemma deals with the critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$.

Lemma 2.14. *For all α and ε such that $0 < |\alpha| \ll \varepsilon \ll 1$, one has :*

- *at all correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ with $f > 0$, $\nabla f_{|g^{-1}(\alpha)}$ points outwards,*
- *at all correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ with $f < 0$, $\nabla f_{|g^{-1}(\alpha)}$ points inwards,*
- *there are no correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ in $f^{-1}(0)$.*

Similarly, we have :

Lemma 2.15. *For ε sufficiently small, one has :*

- *at all correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ with $f > 0$, ∇f points outwards,*
- *at all correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ with $f < 0$, ∇f points inwards,*
- *there are no correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ in $f^{-1}(0)$.*

Lemma 2.16. *We can choose α small enough and we can pertube g into $\tilde{g}$ in such a way that $f_{|W^{\varepsilon}_{\tilde{g}-\alpha}}$ has only Morse critical points.*

Proof. With the method of Lemma 2.7, we can prove that there exists a small perturbation $\tilde{g}$ of g such that $f_{|W^{\varepsilon}_{\tilde{g}-\alpha}}$ has only Morse critical points outside $\{f = 0\}$. But Lemma 2.2 states that $(0, \alpha)$ is a regular value of $(f, \tilde{g})$ for α small enough. $\square$

Proof of Theorem 2.9. The case n even is proved as in Theorem 2.1. So let us assume that n is odd. Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be the distance function to the origin. Let $\varepsilon > 0$ be sufficiently small so that $f_{|g^{-1}(0)\setminus\{0\}}$ has no critical point in $g^{-1}(0) \setminus \{0\} \cap \{\omega < \varepsilon\}$. Let (δ, α) be a regular value of (f, g) such that :

(1) $0 < |\alpha| \ll |\delta| \ll \varepsilon$,
(2) the critical points of $f_{|g^{-1}(\alpha)}$ lie in $\{|f| < \delta\} \cap \{\omega < \frac{\epsilon}{2}\}$,
(3) $\{g?0\} \cap W^{\varepsilon}_{f-\delta}$ is diffeomorphic to $\{g?\alpha\} \cap W^{\varepsilon}_{f-\delta}$, where $? \in \{\leq, =, \geq\}$.

Thanks to the three previous lemmas, we can assume as in Theorem 2.1 that we are in a good situation to apply Morse theory for manifolds with corners. Let us assume that $\delta > 0$. By Morse Theory, we obtain :

$$\chi(\{g \geq \alpha\} \cap \{f \geq \delta\} \cap B^n_{\varepsilon}) - \chi(\{g \geq \alpha\} \cap W^{\varepsilon}_{f-\delta}) = 0, \tag{1}$$

$$\chi(\{g \geq \alpha\} \cap \{f \leq \delta\} \cap B^n_{\varepsilon}) - \chi(\{g \geq \alpha\} \cap W^{\varepsilon}_{f-\delta}) = \sum_{i|\mu(p_i)<0} (-1)^{\lambda(p_i)}, \tag{2}$$

$$\chi(\{g \leq \alpha\} \cap \{f \geq \delta\} \cap B^n_{\varepsilon}) - \chi(\{g \leq \alpha\} \cap W^{\varepsilon}_{f-\delta}) = 0, \tag{3}$$

$$\chi(\{g \leq \alpha\} \cap \{f \leq \delta\} \cap B^n_{\varepsilon}) - \chi(\{g \leq \alpha\} \cap W^{\varepsilon}_{f-\delta}) = -\deg_0 \nabla f + \sum_{i|\mu(p_i)>0} (-1)^{\lambda(p_i)}. \tag{4}$$

In the equality (4), the terms $-\deg_0 \nabla f$ appears because we can pertube f in such a way that its critical points lie in $\{|g| \leq \alpha\} \cap \{|f| \leq \delta\}$. The combination $(1) + (2) - (3) - (4)$ together with the Mayer-Vietoris sequence gives :

$$\chi(\{g \geq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \geq \alpha\} \cap W_{f-\delta}^\varepsilon) \\ +\chi(\{g \leq \alpha\} \cap W_{f-\delta}^\varepsilon) = -\sum_i \mathrm{sign}\mu(p_i) \cdot (-1)^{\lambda(p_i)} + \deg_0 \nabla f.$$

We have already seen that $\deg_0 \nabla f = 0$. Moreover, by the remark after Theorem 3.2 in [Du3], we have :

$$\chi(\{g \geq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap B_\varepsilon^n) = \deg_0 \nabla g.$$

Using Lemma 2.13, we find that :

$$\chi(\{g \geq 0\} \cap W_{f-\delta}^\varepsilon) - \chi(\{g \leq 0\} \cap W_{f-\delta}^\varepsilon) = \deg_0 \nabla g + \deg_0 l(f,g).$$

The proof for δ negative is obtained replacing f with $-f$. $\square$

3. A generalization of Fukui's formula

In this section, we present a generalization of Fukui's formula mentionned in the introduction. As in the previous section, we need to put a condition. More precisely, let $(x_0, x_1, \ldots, x_n)$ be a coordinate system in $\mathbb{R}^{1+n}$ and let $F : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. We assume that F satisfies the following condition (Q) : there exists C^∞ vector fields $V_2, \ldots, V_n$ on $\mathbb{R}^{1+n}$ such that $V_2(p), \ldots, V_n(p)$ span the tangent space at p to $F^{-1}(F(p)) \cap x_0^{-1}(x_0(p))$ whenever p is a regular point of (F, x_0) and such that $(e_0, \nabla F(p), V_2(p), \ldots, V_n(p))$ agrees with the orientation of $\mathbb{R}^{1+n}$. Here e_0 is the vector $(1, 0, \ldots, 0)$.

Let $G : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}, 0)$ be an other analytic function-germ. We define three mappings $H(F,G)$, $I(F,G)$ and $J(F,G) : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}^{1+n}, 0)$ by :

$$H(F,G) = (F, G, V_2 G, \ldots, V_n G),$$
$$I(F,G) = (x_0, G, V_2 G, \ldots, V_n G),$$
$$J(F,G) = (x_0 F, G, V_2 G, \ldots, V_n G).$$

Our first aim is to prove the following theorem :

Theorem 3.1. *If F has an isolated critical point at the origin, 0 is isolated in $J(F,G)^{-1}(0)$ and $(0, \delta, 0)$ is a regular value of (F, G, x_0), then 0 is isolated in $H(F,G)^{-1}(0)$ and we have :*

$$\deg_0 H(F,G) = \mathrm{sign}(-\delta)^n \cdot \Big[\chi\left(W_{(F,G-\delta)}^\varepsilon \cap \{x_0 \geq 0\}\right) \\ -\chi\left(W_{(F,G-\delta)}^\varepsilon \cap \{x_0 \leq 0\}\right)\Big],$$

$$\deg_0 J(F,G) = \mathrm{sign}(-\delta)^n \cdot \left[\chi\left(W_{(F,G-\delta)}^\varepsilon\right) - \chi\left(W_{(G-\delta,x_0)}^\varepsilon\right)\right],$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

To establish this theorem, we need a series of lemmas. From now on, we will assume that the three assumptions of Theorem 3.1 are fullfilled. For all $(i,j) \in \{1,\ldots,n\}^2$, we will set $M_{ij} = \frac{\partial(F,G)}{\partial(x_i,x_j)}$.

Lemma 3.2. *For $\delta \neq 0$ sufficiently small, $(0,\delta)$ is a regular value of (F,G).*

Lemma 3.3. *The origin is an isolated singularity of $F^{-1}(0) \cap G^{-1}(0)$.*

Lemma 3.4. *Let $\delta \neq 0$ be sufficiently small so that $F^{-1}(0) \cap G^{-1}(\delta)$ is smooth submanifold (or empty) of codimension 2 near the origin. Let p be a point in $F^{-1}(0) \cap G^{-1}(\delta)$. The function $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ has a critical point at p if and only if $H(F,G)(p) = (0,\delta,0,\ldots,0)$.*

Proof. The function $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ has a critical point at p if and only if $F(p) = 0$, $G(p) = \delta$ and

$$\operatorname{rank}\begin{bmatrix} 1 & 0 & \ldots & 0 \\ F_{x_0}(p) & F_{x_1}(p) & \ldots & F_{x_n}(p) \\ G_{x_0}(p) & G_{x_1}(p) & \ldots & G_{x_n}(p) \end{bmatrix} < 3.$$

First let us suppose that p is a critical point of $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ and remark that necessarly $x_0(p) \neq 0$ because $(0,\delta,0)$ is a regular value of (F,G,x_0). This implies that p is a regular point of (F,x_0) for the critical points of $x_{0|F^{-1}(0)\setminus\{0\}}$ lie in $\{x_0 = 0\}$ by the Curve Selection Lemma and, so, $(V_2(p),\ldots,V_n(p))$ is a basis of $T_p[F^{-1}(0) \cap x_0^{-1}(x_0(p))]$. Since $\nabla G(p)$ belongs to the normal space at p to $F^{-1}(0) \cap x_0^{-1}(x_0(p))$, we find that for each $i \in \{2,\ldots,n\}$, $\langle V_i(p), \nabla G(p)\rangle = 0$.

Let us show the inverse implication. Let p be such that $H(F,G)(p) = (0,\delta,0,\ldots,0)$. If (F,x_0) is not regular at p then $x_0(p) = 0$ and $(0,\delta,0)$ is not a regular value of (F,G,x_0), which is impossible. Hence $(V_2(p),\ldots,V_n(p))$ is a basis of $T_p[F^{-1}(0) \cap x_0^{-1}(x_0(p))]$ and $\nabla G(p)$ is normal to this last tangent space. □

Lemma 3.5. *Under the assumptions of Lemma 3.4, $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ has a non-degenerate critical point at p if and only if $H(F,G)(p) = (0,\delta,0,\ldots,0)$ and $\det DH(F,G)(p) \neq 0$. Furthermore if $\lambda(p)$ is the Morse index of this function at p then :*

$$(-1)^{\lambda(p)} = (-1)^n \cdot \operatorname{sign}\left[\left(\frac{G(p)}{x_0(p)}\right)^n \cdot \det DH(F,G)(p)\right].$$

Proof. First observe that, since $(0,\delta)$ is a regular value of (F,G) and the M_{ij}'s, $i,j \in \{1,\ldots,n\}$, vanish at p, there exists $k \in \{1,\ldots,n\}$ such that $\frac{\partial(F,G)}{\partial(x_0,x_k)}(p) \neq 0$. Assume that $k = 1$. This implies that $F_{x_1}(p) \neq 0$ for otherwise $G_{x_1}(p) \neq 0$ and $F_{x_j}(p) = 0$ for $j \in \{2,\ldots,n\}$, which means that p is not a regular point of (F,x_0) and $x_0(p) = 0$.

From [Sz2,p349-350], p is a Morse critical point of $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ if and only if

$$\det\begin{bmatrix} \nabla F(p) \\ \nabla G(p) \\ \nabla N_i(p) \end{bmatrix}_{2\leq i\leq n} \neq 0,$$

where $N_i = \frac{\partial(x_0,F,G)}{\partial(x_0,x_1,x_i)} = M_{1i}$. Moreover, one has :

$$(-1)^{\lambda(p)} = \text{sign}\left(\det\begin{bmatrix} \nabla F(p) \\ \nabla G(p) \\ \nabla M_{1i}(p) \end{bmatrix} \cdot \frac{\partial(F,G)}{\partial(x_0,x_1)}(p)^n\right).$$

Let us relate $\det(\nabla F(p), \nabla G(p), \nabla M_{1i}(p))$ to $\det(\nabla F(p), \nabla G(p), \nabla V_i G(p))$. For $i \in \{2,\ldots,n\}$, let $U_i(p)$ be the vector :

$$(0, F_{x_i}(p), 0, \ldots, 0, -F_{x_1}(p), 0, \ldots, 0),$$

where $-F_{x_1}(p)$ is the $(i+1)$-th coordinate. Then $(U_2(p),\ldots,U_n(p))$ is a basis $T_p[F^{-1}(0)\cap x_0^{-1}(x_0(p))]$ and

$$\det(e_0, \nabla F(p), U_2(p), \ldots, U_n(p)) = (-1)^{n-1}\cdot F_{x_1}(p)^{n-2}\cdot\left(\sum_{i=1}^n F_{x_i}(p)^2\right).$$

Hence there exists a $(n-1)\times(n-1)$-matrix $B(p)$ such that :

$$\begin{pmatrix} e_0 \\ \nabla F(p) \\ U_i(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B(p) \end{pmatrix}\cdot\begin{pmatrix} e_0 \\ \nabla F(p) \\ V_i(p) \end{pmatrix},$$

with $\text{sign}[\det B(p)] = (-1)^{n-1}\cdot\text{sign}[F_{x_1}(p)^{n-2}]$. As we proceed in Lemma 2.5, we have :

$$\text{sign}\left[\det\left(e_0, \nabla F(p), \nabla U_i G(p)\right)\right] = \\ (-1)^{n-1}\cdot\text{sign}\left[F_{x_1}(p)^{n-2}\cdot\det\left(e_0,\nabla F(p),\nabla V_i G(p)\right)\right].$$

Since e_0 is a linear combination of $\nabla F(p)$ and $\nabla G(p)$, it is easy to see that :

$$\text{sign}\left[\det\left(\nabla F(p), \nabla G(p), \nabla U_i G(p)\right)\right] = \\ (-1)^{n-1}\cdot\text{sign}\left[F_{x_1}(p)^{n-2}\cdot\det\left(\nabla F(p),\nabla G(p),\nabla V_i G(p)\right)\right].$$

Using the fact that $U_i G(p) = -M_{1i}(p)$, we find that :

$$(-1)^{\lambda(p)} = \text{sign}\left(\det\begin{bmatrix} \nabla F(p) \\ \nabla G(p) \\ \nabla V_i G(p) \end{bmatrix} \cdot \frac{\partial(F,G)}{\partial(x_0,x_1)}(p)^n \cdot F_{x_1}(p)^{n-2}\right).$$

It remains to study the sign of $\frac{\partial(F,G)}{\partial(x_0,x_1)}(p)$. By the Curve Selection Lemma, we can assume that p is on the image of an analytic arc $\gamma :]0,\varepsilon[\to F^{-1}(0)$ such that $M_{ij}(\gamma(t)) = 0$ for $t \in]0,\varepsilon[$ and $(i,j)\in\{1,\ldots,n\}^2$. One has $\sum_{i=1}^n F_{x_i}(\gamma)\cdot\gamma_i' = 0$ since $F\circ\gamma = 0$ and $(G\circ\gamma)' = \sum_{i=1}^n G_{x_i}(\gamma)\cdot\gamma_i'$.

Multiplying the first equality by G_{x_1}, the second by F_{x_1} and making the difference leads to :

$$F_{x_0}G_{x_1} - G_{x_0}F_{x_1} = -\frac{(G \circ \gamma)'}{\gamma_0'} \cdot F_{x_1}.$$

Hence if $\delta \neq 0$ is small enough, $\text{sign}\left(\frac{\partial(F,G)}{\partial(x_0,x_1)}^n \cdot F_{x_1}^{n-2}\right) = -\text{sign}\left(\frac{G}{x_0}\right)$ at p. $\square$

The following lemma deals with the critical points of $x_{0|\partial W^\varepsilon_{(F,G-\delta)}}$.

Lemma 3.6. *Assume that $(0,\delta,0)$ is a regular value of (F,G,x_0) for δ sufficiently small. Then, for ε and δ such that $0 < |\delta| \ll \varepsilon \ll 1$:*

- *the vector $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ points outwards at all correct critical points of $x_{0|\partial W^\varepsilon_{(F,G-\delta)}}$ with $x_0 > 0$,*
- *the vector $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ points inwards at all correct critical points of $x_{0|\partial W^\varepsilon_{(F,G-\delta)}}$ with $x_0 < 0$,*
- *there are no correct critical points of $x_{0|\partial W^\varepsilon_{(F,G-\delta)}}$ in $\{x_0 = 0\}$.*

Proof. The proof is the same as [Du1], Lemma 4.1. $\square$

Lemma 3.7. *If for δ small enough $(0,\delta,0)$ is a regular value of (F,G,x_0), then we can perturbe G into $\tilde{G}$ in such a way that $x_{0|W^\varepsilon_{(F,\tilde{G}-\delta)}}$ has only Morse critical points in $W^\varepsilon_{(F,\tilde{G}-\delta)} \setminus \{x_0 = 0\}$.*

Proof. The proof is similar to the proofs of Lemma 2.7 and Lemma 4.2 in [Du1]. Let us describe it briefly. Let $(x_0,\ldots,x_n;t_1,\ldots,t_n) = (x;t)$ be a coordinate system of $\mathbb{R}^{2n+1}$ and let

$$\bar{G}(x,t) = G(x) + t_1x_1 + \cdots + t_nx_n.$$

For $(i,j) \in \{1,\ldots,n\}^2$, we define $M_{ij}(x,t)$ by $M_{ij}(x,t) = \frac{\partial(F,\bar{G})}{\partial(x_i,x_j)}$. Note that $M_{ij}(x,t) = m_{ij}(x) + F_{x_i}t_j - F_{x_j}t_i$. Let Γ be defined by :

$$\Gamma = \left\{(x,t) \in \mathbb{R}^{2n+1} \mid F(x) = 0 \text{ and } M_{ij}(x,t) = 0 \text{ for } (i,j) \in \{1,\ldots,n\}^2\right\}.$$

In the same way as in Lemma 2.7 and Lemma 4.2, we can prove that $\Gamma \setminus \{x_0 = 0\}$ is a smooth manifold (or empty) of dimension $n+1$. Then we conclude with the following mapping :

$$\begin{array}{cccc} \pi & : & \Gamma \setminus \{x_0 = 0\} & \rightarrow & \mathbb{R}^{1+n} \\ & & (x,t) & \mapsto & (\bar{G}(x,t),t). \end{array}$$

$\square$

Lemma 3.8. *The function $G_{|\{x_0=0\}}$ has an isolated critical point at the origin.*

Proof. Since $J(F,G)$ has an isolated zero at 0, the point $(0,0,0)$ is isolated in $I(F,G)^{-1}(0)$. This would not be the case if 0 in $\mathbb{R}^n$ was not an isolated critical point of $G_{|\{x_0=0\}}$. $\square$

Lemma 3.9. *Let $\delta \neq 0$ be sufficiently small so that $\{x_0 = 0\} \cap G^{-1}(\delta)$ is a smooth submanifold of codimension 2 (or empty) near the origin. Let s be a point in $\{x_0 = 0\} \cap G^{-1}(\delta)$. The function $F_{|\{x_0=0\}\cap G^{-1}(\delta)}$ has a critical point at s if and only if $I(F,G)(s) = (0,\delta,0,\dots,0)$.*

Proof. Since $(0,\delta,0)$ is a regular value of (F,G,x_0), we can apply the proof of Lemma 2.11. $\square$

Lemma 3.10. *Under the assumptions of Lemma 3.9, $F_{|G^{-1}(\delta)\cap x_0^{-1}(0)}$ has a non-degenerate critical point at s if and only if $I(F,G)(s) = (0,\delta,0,\dots,0)$ and $\det DI(F,G)(s) \neq 0$. Furthermore if $\mu(s)$ is the Morse index of this function at s then :*

$$(-1)^{\mu(s)} = (-1)^{n-1} \cdot \operatorname{sign}\left[\left(\frac{G(s)}{F(s)}\right)^n \cdot \det DI(F,G)(s)\right].$$

Proof. The proof is the same as Lemmas 2.5, 2.13 and 3.5. We leave it to the reader. $\square$

The following lemma deals with the critical points of $F_{|\partial W^{\varepsilon}_{(G-\delta,x_0)}}$.

Lemma 3.11. *Assume that $(0,\delta,0)$ is a regular value of (F,G,x_0) for δ sufficiently small. Then, for ε such that $0 < |\delta| \ll \varepsilon \ll 1$:*

- *the vector $\nabla F_{|x_0^{-1}(0)\cap G^{-1}(\delta)}$ points outwards at all correct critical points of $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ with $F > 0$,*
- *the vector $\nabla F_{|x_0^{-1}(0)\cap G^{-1}(\delta)}$ points inwards at all correct critical points of $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ with $F < 0$,*
- *there are no correct critical points of $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ in $F^{-1}(0)$.*

Lemma 3.12. *We can perturb G into $\tilde{G}$ in such a way that $F_{|W^{\varepsilon}_{(\tilde{G}-\delta,x_0)}}$ has only Morse critical point.*

Proof. The same method as Lemma 2.16 can be applied, because we have assumed that $(0,\delta,0)$ is a regular value of (F,G,x_0). $\square$

Proof of Theorem 3.1. It is easy to see that 0 is isolated in $H(F,G)^{-1}(0)$ and $I(F,G)^{-1}(0)$. Let $\omega : \mathbb{R}^{1+n} \to \mathbb{R}$ be the distance function to the origin. Because 0 is isolated in $H(F,G)^{-1}(0)$, $x_{0|F^{-1}(0)\cap G^{-1}(0)\setminus\{0\}}$ has no critical point and then, choosing δ sufficiently small, we can assume that $x_{0|F^{-1}(0)\cap G^{-1}(\delta)\cap\{\omega<\varepsilon\}}$ admits its critical points in $W^{\varepsilon/4}_{(F,G-\delta)}$. Thus the critical points of $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$ are correct. By Lemmas 3.6 and 3.7, we can suppose that $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ is a correct Morse function, that its critical points lie in $B_{\varepsilon/2}$, that at the correct critical points of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ lying in $\{x_0 > 0\}$ (resp. in $\{x_0 < 0\}$), $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ points outwards (resp. inwards) and that there are no correct critical points of $x_{0|F_\delta}$ in $\{x_0 = 0\}$. Applying Morse Theory for manifolds with boundary, we find :

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}, W^{\varepsilon}_{(F,G-\delta,x_0)}) = \sum_{i | x_0(p_i)>0} (-1)^{\lambda(p_i)},$$

where $\{p_i\}$ is the set of Morse critical points of $x_{0|W^\varepsilon_{(F,G-\delta)}}$. Similarly, we have :

$$\chi(W^\varepsilon_{(F,G-\delta)}\{x_0 \leq 0\}, W^\varepsilon_{(F,G-\delta,x_0)}) = (-1)^{n-1} \sum_{i|x_0(p_i)<0} (-1)^{\lambda(p_i)}.$$

By Lemma 3.4, p is a critical point of $x_{0|W^\varepsilon_{(F,G-\delta)}}$ if and only if

$$H(F,G)(p) = (0,\delta,0,\dots,0).$$

Hence $H(F,G)^{-1}(0,\delta,0,\dots,0)$ is the set of critical points of $x_{0|W^\varepsilon_{(F,G-\delta)}}$. Since $x_{0|W^\varepsilon_{(F,G-\delta)}}$ is a Morse function, $\det DH(F,G)(p) \neq 0$ for each p in $H^{-1}(0,\delta,0,\dots,0)$ by Lemma 3.5. Hence $(0,\delta,0,\dots,0)$ is a regular value of $H(F,G)$ and

$$\deg_0 H = \sum_{p\in H^{-1}(0,\delta,0,\dots,0)} \operatorname{sign}\left[\det DH(F,G)(p)\right].$$

Combining this with the above equalities and Lemma 3.5, we obtain the first equality. Let us study the critical points of $F_{|W^\varepsilon_{(G-\delta,x_0)}}$. Thanks to Lemmas 3.11 and 3.12, we can assume that we are in a good situation to apply Morse theory. We have :

$$\chi(W^\varepsilon_{(G-\delta,x_0)} \cap \{F \geq 0\}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}) = \sum_{j|F(s_j)>0} (-1)^{\mu(s_j)},$$

where $\{s_j\}$ is the set of Morse critical points of $F_{|W^\varepsilon_{(G-\delta,x_0)}}$. Similarly, we have :

$$\chi(W^\varepsilon_{(G-\delta,x_0)} \cap \{F \leq 0\}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}) = (-1)^{n-1} \sum_{j|F(s_j)>0} (-1)^{\mu(s_j)}.$$

Hence, we get :

$$\chi(W^\varepsilon_{(G-\delta,x_0)}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}) = \sum_{j|F(s_j)>0} (-1)^{\mu(s_j)} + (-1)^{n-1} \cdot \sum_{j|F(s_j)<0} (-1)^{\mu(s_j)}.$$

Applying Lemma 3.10, this gives :

$$\chi(W^\varepsilon_{(G-\delta,x_0)}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}) = -\operatorname{sign}(-\delta)^n \cdot \sum_j \operatorname{sign}[F(s_j)] \cdot \det DI(F,G)(s_j).$$

Similarly, we have :

$$\chi(W^\varepsilon_{(F,G-\delta)}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}) = \operatorname{sign}(-\delta)^n \cdot \sum_i \operatorname{sign}[x_0(p_i)] \cdot \det DH(F,G)(p_i).$$

But the sets $\{p_i\}$ and $\{s_j\}$ are exactly the preimages of $(0,\delta,0,\ldots,0)$ by $J(F,G)$. Furthermore, each p_i is a regular point of $J(F,G)$ and

$$\text{sign}\,[\text{det}DJ(F,G)(p_i)] = \text{sign}\,[x_0(p_i)\cdot \text{det}DH(F,G)(p_i)]\,.$$

Each s_j is a regular value of $J(F,G)$ as well and

$$\text{sign}\,[\text{det}DJ(F,G)(s_j)] = \text{sign}\,[F(s_j)\cdot \text{det}DI(F,G)(s_j)]\,.$$

With all these informations, it is easy to conclude. $\square$

We want to apply these formulas when F and G are one-parameter deformations of two germs f and g. First let us define the function-germ $F_0 : (\mathbb{R}^n,0)\to(\mathbb{R},0)$ by $F_0(x_1,\ldots,x_n)=F(0,x_1,\ldots,x_n)$.

Lemma 3.13. *Assume that the function-germ F_0 has an isolated critical point at the origin.. Then for δ sufficiently small, $(0,\delta,0)$ is a regular of (F,G,x_0). Let us suppose that $\delta>0$, then for $0<\delta\ll\varepsilon\ll 1$, one has :*

$$W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\geq 0\}\simeq \partial W^{\varepsilon}_F\cap\{G\geq 0\}\cap\{x_0\geq 0\}\simeq W^{\varepsilon}_{(F,x_0-\delta)}\cap\{G\geq 0\},$$
$$W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\leq 0\}\simeq \partial W^{\varepsilon}_F\cap\{G\geq 0\}\cap\{x_0\leq 0\}\simeq W^{\varepsilon}_{(F,x_0+\delta)}\cap\{G\geq 0\},$$
$$W^{\varepsilon}_{(F,G+\delta)}\cap\{x_0\geq 0\}\simeq \partial W^{\varepsilon}_F\cap\{G\leq 0\}\cap\{x_0\geq 0\}\simeq W^{\varepsilon}_{(F,x_0-\delta)}\cap\{G\leq 0\},$$
$$W^{\varepsilon}_{(F,G+\delta)}\cap\{x_0\leq 0\}\simeq \partial W^{\varepsilon}_F\cap\{G\leq 0\}\cap\{x_0\leq 0\}\simeq W^{\varepsilon}_{(F,x_0+\delta)}\cap\{G\leq 0\},$$

where $\simeq$ means diffeomorphic to.

Proof. Let us prove the first line. It is just an adaptation to our case of the deformation argument given by Milnor [Mi,Lemma 11.3]. We can construct a vector field v_1 on $W^{\varepsilon}_F\setminus\{G=0\}$ such that $\langle v_1(x),\nabla G(x)\rangle$ and $\langle v_1(x),x\rangle$ are both positive. Similarly there exists a vector field v_2 on $W^{\varepsilon}_{(F,x_0)}\setminus\{G=0\}$ such that $\langle v_2(x),\nabla G(x)\rangle$ and $\langle v_2(x),x\rangle$ are both positive. Using a collar, we can extend v_2 to a vector field $\tilde{v_2}$ defined in a neighborhood of $W^{\varepsilon}_{(F,x_0)}\setminus\{G=0\}$ in $W^{\varepsilon}_F\cap\{x_0\geq 0\}\setminus\{G=0\}$ such that $\langle\tilde{v}_2(x),\nabla G(x)\rangle$ and $\langle\tilde{v}_2(x),x\rangle$ are positive. Gluing v_1 and $\tilde{v_2}$ gives a new vector field w on $W^{\varepsilon}_F\cap\{x_0\geq 0\}\setminus\{G=0\}$. The diffeomorphism between $W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\geq 0\}$ and $\partial W^{\varepsilon}_F\cap\{G\geq 0\}\cap\{x_0\geq 0\}$ is obtained integrating the trajectories of w. Similarly $W^{\varepsilon}_{(F,x_0-\delta)}\cap\{G\geq 0\}$ is diffeomorphic to $\partial W^{\varepsilon}_F\cap\{G\geq 0\}\cap\{x_0\geq 0\}$ because, by Lemma 3.3, $F^{-1}(0)\cap G^{-1}(0)$ is smooth outside the origin. $\square$

We want to compute $\chi(W^{\varepsilon}_{(F,G,x_0-\delta)})$. By the Mayer-Vietoris sequence, we know that :

$$\chi(W^{\varepsilon}_{(F,x_0-\delta)})=\chi(W^{\varepsilon}_{(F,x_0-\delta)}\cap\{G\geq 0\})+ \chi(W^{\varepsilon}_{(F,x_0-\delta)}\cap\{G\leq 0\})-\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}).$$

Hence, by Lemma 3.13, we find that if $\delta>0$, then :

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)})=\chi(W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\geq 0\})+ \chi(W^{\varepsilon}_{(F,G+\delta)}\cap\{x_0\geq 0\})-\chi(W^{\varepsilon}_{(F,x_0-\delta)}),$$

$$\chi(W^{\varepsilon}_{(F,G,x_0+\delta)})=\chi(W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\leq 0\})+$$

$$\chi(W^\varepsilon_{(F,G+\delta)} \cap \{x_0 \leq 0\}) - \chi(W^\varepsilon_{(F,x_0+\delta)}).$$

The Euler-Poincaré characteristic of $W^\varepsilon_{(F,x_0\pm\delta)}$ can be computed thanks to formulas established in [Fu], as explained in [Du3, Theorem 3.2]. More precisely, let $L(F) : (\mathbb{R}^{1+n}, 0) \to (\mathbb{R}^{1+n}, 0)$ be the mapping defined by $L(F) = (F, F_{x_1}, \ldots, F_{x_n})$. If $L(F)$ and ∇F_0 have an isolated zero at the origin, then ∇F has an isolated zero at the origin and the following theorem explains how to compute $W^\varepsilon_{(F,x_0\pm\delta)}$.

Theorem 3.14. *Let δ and ε be such that $0 < |\delta| \ll \varepsilon \ll 1$. If n is even then :*

$$\chi(W^\varepsilon_{(F,x_0-\delta)}) = 1 - \deg_0 \nabla F_0.$$

If n is odd then :

$$\chi(W^\varepsilon_{(F,x_0-\delta)}) = 1 - \deg_0 \nabla F - \operatorname{sign}(\delta) \cdot \deg_0 L(F).$$

Proof. See [Fu] and [Du3]. □

At this point, we have assumed that :

(1) F has an isolated critical point at the origin,
(2) $J(F,G)$ has an isolated zero at the origin,
(3) $(0, \delta, 0)$ is a regular value of (F, G, x_0),
(4) F_0 has an isolated critical point at the origin.

By the Curve Selection Lemma, the assumption (4) implies the assumption (3). Moreover, it means that 0 is isolated in $\{F = F_{x_1} = \cdots = F_{x_n} = x_0 = 0\}$. Since this last set is equal to $\{F = F_{x_1} = \cdots = F_{x_n} = 0\}$ near the origin thanks to the Curve Selection Lemma and the fact that $F^{-1}(0)$ has an isolated singularity, we have that (4) implies that 0 is isolated in $L(F)^{-1}(0)$. So, under the assumption (4), we can apply the above theorem.

It remains to compute $\chi(W^\varepsilon_{(F,G\pm\delta)} \cap \{x_0 ? 0\})$, $? \in \{\leq, \geq\}$. By the Mayer-Vietoris sequence, we have :

$$\chi(W^\varepsilon_{(F,G-\delta)}) = \chi(W^\varepsilon_{(F,G-\delta)} \cap \{x_0 \geq 0\}) + \\ \chi(W^\varepsilon_{(F,G-\delta)} \cap \{x_0 \leq 0\}) - \chi(W^\varepsilon_{(F,G-\delta,x_0)}).$$

But Theorem 2.1 enables us to compute $\chi(W^\varepsilon_{(F,G-\delta)})$ and $\chi(W^\varepsilon_{(F,G-\delta,x_0)})$. Let $G_0 : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ be defined by $G_0(x_1, \ldots, x_n) = G(0, x_1, \ldots, x_n)$ and let us assume that it has an isolated critical point at the origin. Then using Theorem 3.1 and Khimshiashvili's formula, we find that :

$$\chi(W^\varepsilon_{(F,G-\delta)}) = 1 + \operatorname{sign}(-\delta)^n \cdot [\deg_0 J(F,G) - \deg_0 \nabla G_0].$$

Now observe that, since F satisfies the condition (Q), F_0 satisfies the condition (P) of Section 2 with the vector fields $V^0_2, \ldots, V^0_n$ given by :

$$V^0_i(x_1, \ldots, x_n) = V_i(0, x_1, \ldots, x_n).$$

Let $k(F_0, G_0) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be defined by :

$$k(F_0, G_0) = (F_0, V^0_2 G_0, \ldots, V^0_n G_0).$$

By Theorem 2.1, we have :

$$\text{if } n \text{ is even } : \chi(W^{\varepsilon}_{(F,G-\delta,x_0)}) = 1 - \deg_0 \nabla F_0 + \text{sign}(\delta)\cdot \deg_0 k(F_0,G_0),$$
$$\text{if } n \text{ is odd } : \chi(W^{\varepsilon}_{(F,G-\delta,x_0)}) = 1 - \deg_0 k(F_0,G_0).$$

Let us focus first on the case n even. We have :

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) = \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) =$$
$$2 + \deg_0 J(F,G) - \deg_0 \nabla G_0 - \deg_0 \nabla F_0 + \text{sign}(\delta)\cdot \deg_0 k(F_0,G_0).$$

This gives :

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) = 1 + \frac{1}{2}\left[\deg_0 J(F,G) - \deg_0 \nabla G_0 \right.$$
$$\left. -\deg_0 \nabla F_0 + \text{sign}(\delta)\cdot \deg_0 k(F_0,G_0) + \deg_0 H(F,G)\right],$$
$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) = 1 + \frac{1}{2}\left[\deg_0 J(F,G) - \deg_0 \nabla G_0 \right.$$
$$\left. -\deg_0 \nabla F_0 + \text{sign}(\delta)\cdot \deg_0 k(F_0,G_0) - \deg_0 H(F,G)\right].$$

Collecting all these informations, we obtain :

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = 1 + \deg_0 J(F,G) - \deg_0 \nabla G_0 + \text{sign}(\delta)\cdot \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\}) - \chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \leq 0\}) = \deg_0 k(F_0,G_0).$$

If n is odd, we have :

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) = -\text{sign}(\delta)\cdot \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) =$$
$$2 - \text{sign}(\delta)\cdot [\deg_0 J(F,G) - \deg_0 \nabla G_0] - \deg_0 k(F_0,G_0).$$

This gives :

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) = 1 - \frac{1}{2}\left[\text{sign}(\delta)\cdot (\deg_0 J(F,G) - \deg_0 \nabla G_0 \right.$$
$$\left. +\deg_0 H(F,G)) + \deg_0 k(F_0,G_0)\right],$$
$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) = 1 - \frac{1}{2}\left[\text{sign}(\delta)\cdot (\deg_0 J(F,G) - \deg_0 \nabla G_0 \right.$$
$$\left. -\deg_0 H(F,G)) + \deg_0 k(F_0,G_0)\right].$$

Finally we find :

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = 1 - \deg_0 k(F_0,G_0) - \deg_0 \nabla F - \text{sign}(\delta)\cdot \deg_0 L(F),$$
$$\chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\}) - \chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \leq 0\}) =$$
$$-\deg_0 J(F,G) + \deg_0 \nabla G_0 - \text{sign}(\delta)\cdot \deg_0 H(F,G).$$

Here, we have to remark that :

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = \frac{1}{2}\chi(\partial W^{\varepsilon}_{(F,G,x_0-\delta)}) = \frac{1}{2}\chi(\partial W^{\varepsilon}_{(F,G,x_0)}) = 1 - \deg_0 k(F_0,G_0),$$

by Corollary 2.8. Hence, we get that $\deg_0 \nabla F = \deg_0 L(F) = 0$.

We can reformulate these results in terms of one-parameter deformations of function-germs. Let $(x_1,\ldots,x_n)$ be a coordinate system of $\mathbb{R}^n$. Let $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a function-germ with an isolated critical point at the origin. Let $g:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be a function-germ with an isolated critical point at the origin such that the mapping $k(f,g):(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ has an isolated zero where $k(f,g)$ is defined as in Section 2. Let $(\lambda,x_1,\ldots,x_n)$ be a coordinate system in $\mathbb{R}^{1+n}$ and let $F:(\mathbb{R}^{1+n},0)\to(\mathbb{R},0)$ (resp. $G:(\mathbb{R}^{1+n},0)\to(\mathbb{R},0)$) be a one-parameter deformation of f (resp g), i.e $F(0,x)=f(x)$ (resp. $G(0,x)=g(x)$). We will use the notations $f_t(x)=F(t,x)$ and $g_t(x)=G(t,x)$. We assume that :

(1) F has an isolated critical point at the origin,
(2) the mapping $J(F,G)$ has an isolated zero at the origin,
(3) F satisfies the condition (Q) (which implies that f satisfies the condition (P)).

We note that F_0 and G_0 have an isolated critical point because $F_0=f$ and $G_0=g$. So we are in situation to apply the above process.

Corollary 3.15. *For t and ε with $0<|t|\ll\varepsilon\ll 1$, we have :*
- if n is odd :

$$\chi(W^{\varepsilon}_{(f_t,g_t)})=1-\deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f_t}\cap\{g_t\geq 0\})-\chi(W^{\varepsilon}_{f_t)}\cap\{g_t\leq 0\})=$$
$$-\deg_0 J(F,G)+\deg_0\nabla g-\mathrm{sign}(t)\cdot\deg_0 H(F,G),$$

- if n is even :

$$\chi(W^{\varepsilon}_{(f_t,g_t)})=1+\deg_0 J(F,G)-\deg_0\nabla g+\mathrm{sign}(t)\cdot\deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f_t}\cap\{g_t\geq 0\})-\chi(W^{\varepsilon}_{f_t}\cap\{g_t\leq 0\})=\deg_0 k(f,g).$$

Let us consider the following deformations of f and g :

$$F(\lambda,x)=f(x)-\gamma_1(\lambda)\text{ and }G(\lambda,x)=g(x)-\gamma_2(\lambda),$$

where $\gamma=(\gamma_1,\gamma_2):(\mathbb{R},0)\to(\mathbb{R}^2,0)$ is an analytic arc such that $\gamma(t)\neq 0$ if $t\neq 0$ and $\gamma_1'(t)\neq 0$ if $t\neq 0$. With this last condition, the function F has an isolated critical point at the origin. Furthermore, we assume that f satisfies the condition (P). This implies that F satisfies the condition (Q) with $V_i(\lambda,x)=v_i(x)$ for $i=2,\ldots,n$. Let us denote by $\mathrm{Disc}(f,g)$ the discriminant of the mapping (f,g). The following lemma tells us when the points in the image of γ are regular value of (f,g) near the origin.

Lemma 3.16. *The origin $(0,0)$ is isolated in $H(F,G)^{-1}(0)$ if and only if 0 is isolated in $\mathrm{Disc}(f,g)\cap\gamma(I)$, where I is a small open interval in $\mathbb{R}$ containing 0.*

Proof. The point $(0,0)$ is isolated in $H(F,G)^{-1}(0)$ if and only if for all $(t,x)\neq(0,0)$ such that $F(t,x)=G(t,x)=0$, there exists $i\in\{2,\ldots,n\}$ such that $v_iG(t,x)\neq 0$. Let us remark that if $x\neq 0$ is such that $F(0,x)=G(0,x)=0$ then $v_iG(0,x)\neq 0$ for some i in $\{2,\ldots,n\}$ because $f^{-1}(0)\cap$

$g^{-1}(0)$ has an isolated singularity. Therefore the point $(0,0)$ is isolated in $H(F,G)^{-1}(0)$ if and only if for all (t,x) with $t \neq 0$ such that $F(t,x) = G(t,x) = 0$ there exists $i \in \{2,\dots,n\}$ such that $v_i G(t,x) \neq 0$. This is equivalent to the fact that for all $t \neq 0$ and for all x such that $f(x) = \gamma_1(t)$ and $g(x) = \gamma_2(t)$, $\nabla f(x)$ and $\nabla g(x)$ are not colinear. $\square$

Corollary 3.15 can be restated in this situation.

Corollary 3.17. *Assume that f and g have an isolated singularity and that $\gamma_1'(t) \neq 0$ if $t \neq 0$. Assume that $J(F,G)$ and $k(f,g)$ have an isolated zero at the origin then for t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have :*
- if n is odd :

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 - \deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = \\ -\deg_0 J(F,G) + \deg_0 \nabla g - \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$

- if n is even :

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 + \deg_0 J(F,G) - \deg_0 \nabla g + \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g_t \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = \deg_0 k(f,g).$$

Let us examine the situation when $\lambda_1(t) = t$ and $\lambda_2(t) = 0$. In this case, we can check that $\deg_0 J(F,G) = 0$ and that $\deg_0 H = -\deg_0 l(f,g)$, where $l(f,g)$ is defined in Section 2. Hence, we recover the results of Theorem 2.9.

4. Explicit formulas

In this section, we present some situations where the conditions (P) and (Q) are satisfied.

4.1. **Case $n = 2$, 4 or 8.** As explained in [FK], when $n = 2$, 4 or 8, the condition (P) is satisfied for any function-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. If ∂_{x_i} denotes the vector $e_i = (0,\dots,0,1,0\dots,0)$ where 1 is the i-th coordinate, then the vectors $v_2,\dots,v_n$ are given by, if $n = 2$:

$$v_2 = -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2},$$

if $n = 4$:

$$v_2 = -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2} - f_{x_4}\partial_{x_3} + f_{x_3}\partial_{x_4},$$
$$v_3 = -f_{x_3}\partial_{x_1} + f_{x_4}\partial_{x_2} + f_{x_1}\partial_{x_3} - f_{x_2}\partial_{x_4},$$
$$v_4 = -f_{x_4}\partial_{x_1} - f_{x_3}\partial_{x_2} + f_{x_2}\partial_{x_3} + f_{x_1}\partial_{x_4},$$

if $n = 8$:

$$v_2 = -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2} - f_{x_4}\partial_{x_3} + f_{x_3}\partial_{x_4} \\ -f_{x_6}\partial_{x_5} + f_{x_5}\partial_{x_6} + f_{x_8}\partial_{x_7} - f_{x_7}\partial_{x_8},$$
$$v_3 = -f_{x_3}\partial_{x_1} + f_{x_4}\partial_{x_2} + f_{x_1}\partial_{x_3} - f_{x_2}\partial_{x_4} \\ -f_{x_7}\partial_{x_5} - f_{x_8}\partial_{x_6} + f_{x_5}\partial_{x_7} + f_{x_6}\partial_{x_8},$$
$$v_4 = -f_{x_4}\partial_{x_1} - f_{x_3}\partial_{x_2} + f_{x_2}\partial_{x_3} + f_{x_1}\partial_{x_4}$$

$$
\begin{aligned}
& \qquad\qquad -f_{x_8}\partial_{x_5} + f_{x_7}\partial_{x_6} - f_{x_6}\partial_{x_7} - f_{x_5}\partial_{x_8},\\
v_5 &= -f_{x_5}\partial_{x_1} + f_{x_6}\partial_{x_2} + f_{x_7}\partial_{x_3} + f_{x_8}\partial_{x_4}\\
& \qquad\qquad +f_{x_1}\partial_{x_5} - f_{x_2}\partial_{x_6} - f_{x_3}\partial_{x_7} - f_{x_4}\partial_{x_8},\\
v_6 &= -f_{x_6}\partial_{x_1} - f_{x_5}\partial_{x_2} + f_{x_8}\partial_{x_3} - f_{x_7}\partial_{x_4}\\
& \qquad\qquad +f_{x_2}\partial_{x_5} + f_{x_1}\partial_{x_6} + f_{x_4}\partial_{x_7} - f_{x_3}\partial_{x_8},\\
v_7 &= -f_{x_7}\partial_{x_1} - f_{x_8}\partial_{x_2} - f_{x_5}\partial_{x_3} + f_{x_6}\partial_{x_4}\\
& \qquad\qquad +f_{x_3}\partial_{x_5} - f_{x_4}\partial_{x_6} + f_{x_1}\partial_{x_7} + f_{x_2}\partial_{x_8},\\
v_8 &= -f_{x_8}\partial_{x_1} + f_{x_7}\partial_{x_2} - f_{x_6}\partial_{x_3} - f_{x_5}\partial_{x_4}\\
& \qquad\qquad +f_{x_4}\partial_{x_5} + f_{x_3}\partial_{x_6} - f_{x_2}\partial_{x_7} + f_{x_1}\partial_{x_8}.
\end{aligned}
$$

The condition (Q) is also fullfilled, the vectors V_i being given by, if $n = 2$:

$$V_2 = -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2},$$

if $n = 4$:

$$
\begin{aligned}
V_2 &= -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2} - F_{x_4}\partial_{x_3} + F_{x_3}\partial_{x_4},\\
V_3 &= -F_{x_3}\partial_{x_1} + F_{x_4}\partial_{x_2} + F_{x_1}\partial_{x_3} - F_{x_2}\partial_{x_4},\\
V_4 &= -F_{x_4}\partial_{x_1} - F_{x_3}\partial_{x_2} + F_{x_2}\partial_{x_3} + F_{x_1}\partial_{x_4},
\end{aligned}
$$

if $n = 8$:

$$
\begin{aligned}
V_2 &= -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2} - F_{x_4}\partial_{x_3} + F_{x_3}\partial_{x_4}\\
& \qquad\qquad -F_{x_6}\partial_{x_5} + F_{x_5}\partial_{x_6} + F_{x_8}\partial_{x_7} - F_{x_7}\partial_{x_8},\\
V_3 &= -F_{x_3}\partial_{x_1} + F_{x_4}\partial_{x_2} + F_{x_1}\partial_{x_3} - F_{x_2}\partial_{x_4}\\
& \qquad\qquad -F_{x_7}\partial_{x_5} - F_{x_8}\partial_{x_6} + F_{x_5}\partial_{x_7} + F_{x_6}\partial_{x_8},\\
V_4 &= -F_{x_4}\partial_{x_1} - F_{x_3}\partial_{x_2} + F_{x_2}\partial_{x_3} + F_{x_1}\partial_{x_4}\\
& \qquad\qquad -F_{x_8}\partial_{x_5} + F_{x_7}\partial_{x_6} - F_{x_6}\partial_{x_7} - F_{x_5}\partial_{x_8},\\
V_5 &= -F_{x_5}\partial_{x_1} + F_{x_6}\partial_{x_2} + F_{x_7}\partial_{x_3} + F_{x_8}\partial_{x_4}\\
& \qquad\qquad +F_{x_1}\partial_{x_5} - F_{x_2}\partial_{x_6} - F_{x_3}\partial_{x_7} - F_{x_4}\partial_{x_8},\\
V_6 &= -F_{x_6}\partial_{x_1} - F_{x_5}\partial_{x_2} + F_{x_8}\partial_{x_3} - F_{x_7}\partial_{x_4}\\
& \qquad\qquad +F_{x_2}\partial_{x_5} + F_{x_1}\partial_{x_6} + F_{x_4}\partial_{x_7} - F_{x_3}\partial_{x_8},\\
V_7 &= -F_{x_7}\partial_{x_1} - F_{x_8}\partial_{x_2} - F_{x_5}\partial_{x_3} + F_{x_6}\partial_{x_4}\\
& \qquad\qquad +F_{x_3}\partial_{x_5} - F_{x_4}\partial_{x_6} + F_{x_1}\partial_{x_7} + F_{x_2}\partial_{x_8},\\
V_8 &= -F_{x_8}\partial_{x_1} + F_{x_7}\partial_{x_2} - F_{x_6}\partial_{x_3} - F_{x_5}\partial_{x_4}\\
& \qquad\qquad +F_{x_4}\partial_{x_5} + F_{x_3}\partial_{x_6} - F_{x_2}\partial_{x_7} + F_{x_1}\partial_{x_8}.
\end{aligned}
$$

So all the results of Section 2 and Section 3 can be applied. Note also that the vector fields v_i and V_i are analytic.

4.2. **Case $f_{x_1} \geq 0$ and $F_{x_1} \geq 0$.** The condition (P) is satisfied for a function-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $f_{x_1} \geq 0$ (see [FK,p151]). The vectors $v_2, \ldots, v_n$ are defined by :

$$v_i = -f_{x_i}\partial_{x_1} - \sum_{j=2}^{n}(f_{x_i}f_{x_j} - \delta_{i,j}T)\partial_{x_j},$$

where $T = f_{x_1} + \sum_{j=2}^{n} f_{x_j}^2$ and $\delta_{i,j}$ is the Kronecker symbol. Here we notice that there is a mistake in the computation of the determinant of the matrix M defined p.151 in [FK]. This determinant is $(-1)^n T^{n-1}\sum_{i=0}^{n} g_{x_i}^2$. That is why our v_i's are the opposite of the v_i's defined by Fukui and Khovanskii.

If $F_{x_1} \geq 0$, the condition (Q) is satisfied with the vectors V_i's defined by :

$$V_i = -F_{x_1}\partial_{x_1} - \sum_{j=2}^{n}(F_{x_i}F_{x_j} - \delta_{i,j}T')\partial_{x_j},$$

where $T' = F_{x_1} + \sum_{j=2}^{n} F_{x_j}^2$. Let us remark that in this situation the computation of $\chi(W^{\varepsilon}_{(F,G-\delta)})$ can be simplified thanks to Theorem 2.1. Actually, the function F satisfies the condition (P) with the following vectors :

$$Z_0 = F_{x_0}\partial_{x_1} + \sum_{j=0 \ | \ j\neq 1}^{n} (F_{x_i}F_{x_j} - \delta_{i,j}S)\partial_{x_j},$$

$$Z_i = -F_{x_i}\partial_{x_1} - \sum_{j=0 \ | \ j\neq 1}^{n} (F_{x_i}F_{x_j} - \delta_{i,j}S)\partial_{x_j}, \ i = 2, \ldots, n,$$

where $S = F_{x_1} + F_{x_0}^2 + \sum_{j=2}^{n} F_{x_j}^2$. Let $K(F,G) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ be defined by :

$$K(F,G) = (F, Z_0G, Z_2G, \ldots, Z_nG).$$

Since $F^{-1}(0) \cap G^{-1}(0)$ has an isolated singularity at the origin (Lemma 3.3) then $K(F,G)$ has an isolated zero at the origin (Lemma 2.4). Hence, by Theorem 2.1 and since $\deg_0 \nabla F = 0$ for $F_{x_1} \geq 0$, we have :

if n is odd : $\chi(W^{\varepsilon}_{(F,G-\delta)}) = 1 + \mathrm{sign}(\delta) \cdot \deg_0 K(F,G)$,

if n is even : $\chi(W^{\varepsilon}_{(F,G-\delta)}) = 1 - \deg_0 K(F,G)$.

So Corollary 3.15 can be rewritten without the assumption that g has an isolated critical point at the origin. Namely, with the obvious assumptions, we obtain :

Corollary 4.1. *For t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have :*
- if n is odd :

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 - \deg_0 k(f,g),$$

$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = +\deg_0 K(F,G) - \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$

- if n is even :

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 - \deg_0 K(F,G) + \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$

$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = \deg_0 k(f,g).$

If the deformation (F,G) of (f,g) is of the form $F(\lambda,x) = f(x) - \gamma_1(\lambda)$, $G(\lambda,x) = f(x) - \gamma_2(\lambda)$, then we just need to suppose that $f_{x_1} \geq 0$. Therefore Corollary 3.17 becomes :

Corollary 4.2. *Assume that f and g have an isolated singularity and that $\gamma_1'(t) \neq 0$ if $t \neq 0$. Assume that $J(F,G)$ and $k(f,g)$ have an isolated zero at the origin then for t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have :*
- if n is odd :

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 - \deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = +\deg_0 K(F,G) - \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$

- if n is even :

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 - \deg_0 K(F,G) + \mathrm{sign}(t) \cdot \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g_t \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = \deg_0 k(f,g).$$

Let us end with an example. Let $f(x_1,x_2,x_3,x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$ and $g(x_1,x_2,x_3,x_4) = x_1x_2 + x_3x_4$. These functions have an isolated critical point at the origin and $\deg_0 \nabla f = -1$ and $\deg_0 \nabla g = 1$. The mappings $k(f,g)$ and $l(f,g)$ of Section 2 are :

$$k(f,g)(x) = (x_1^2 + x_2^2 + x_3^2 - x_4^2, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4),$$
$$l(f,g)(x) = (x_1x_2 + x_3x_4, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4).$$

It is not difficult to see that 0 is an isolated root of $k(f,g)$ and $l(f,g)$. Furthermore, $\deg_0 k(f,g) = 0$ because $k(f,g)^{-1}(0,\beta,0,0) = \emptyset$ if $\beta < 0$. If $\beta < 0$ then $l(f,g)^{-1}(0,\beta,0,0)$ consists of the points $p_1 = (0, \sqrt{-\frac{\beta}{2}}, 0, 0)$ and $p_2 = (0, -\sqrt{-\frac{\beta}{2}}, 0, 0)$. Since $\det[Dl(f,g)(p_i)] > 0$, $\deg_0 l(f,g)$ is equal to 2. By Theorem 2.1 and Theorem 2.9, we get that $\chi(W^{\varepsilon}_{(f,g-\delta)}) = 2$, $\chi(W^{\varepsilon}_{(f-\delta,g)}) = -2$ if $\delta > 0$ and $\chi(W^{\varepsilon}_{(f-\delta,g)}) = 2$ if $\delta < 0$. By Corollary 3.17, we have :

$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g_t \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = 0,$$

for an appropriate analytic arc (γ_1,γ_2).

Let us compute $\chi(W^{\varepsilon}_{(f-t,g-t)})$ using Corollary 3.17. The mappings H and J of Section 3 are given by :

$$H(t,x) = (x_1^2 + x_2^2 + x_3^2 - x_4^2 - t, x_1x_2 + x_3x_4 - t, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4),$$
$$J(t,x) = (t \cdot (x_1^2 + x_2^2 + x_3^2 - x_4^2 - t), x_1x_2 + x_3x_4 - t, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4).$$

Let us search the points (t,x) such that $H(t,x)=0$. If $x_2=0$ then clearly $x_1=x_3=x_4=t=0$. If $x_2\neq 0$ then $x_3=x_4=0$ and :

$$\begin{cases} x_1^2+x_2^2-t=0 \\ x_1x_2-t=0 \\ 2x_1^2-2x_2^2=0 \end{cases}$$

This implies that $t^2=4x_2^4=x_2^4$, which is a contradiction. Hence H admits an isolated zero at the origin. Furthermore $\deg_0 H=0$. To see this, let (t,x) be such that $H(t,x)=(0,0,\beta,0,0)$ where $\beta<0$. Necessarily $x_2\neq 0$ and $x_3=x_4=0$. Hence x_1, x_2 and t satisfy the system :

$$\begin{cases} x_1^2+x_2^2-t=0 \\ x_1x_2-t=0 \\ 2x_1^2-2x_2^2=\beta \end{cases}$$

Putting $\gamma=\frac{\beta}{2}$, we find that $x_1^2=\frac{t+\gamma}{2}$, $x_2^2=\frac{t-\gamma}{2}$ and $t^2=\frac{t^2-\gamma^2}{4}$. This last equality is equivalent to $3t^2=-\gamma^2$, which is impossible.

Let us search the points (t,x) such that $J(t,x)=0$. As above, if $x_2=0$ then $x_1=x_3=x_4=t=0$. If $x_2\neq 0$ then $x_3=x_4=0$ and

$$\begin{cases} t(x_1^2+x_2^2-t)=0 \\ x_1x_2-t=0 \\ 2x_1^2-2x_2^2=0 \end{cases}$$

If $t=0$ then $x_1=x_2=0$, which is a contradiction. The case $x_1^2+x_2^2-t=0$ is also impossible as we have already explained. Hence J admits an isolated zero at the origin. Let $\beta<0$ and let us search the points (t,x) such that $J(t,x)=(\frac{\beta^2}{8},0,\beta,0,0)$. Necessarily $x_2\neq 0$ and $x_3=x_4=0$. Hence x_1, x_2 and t satisfy the system :

$$\begin{cases} t(x_1^2+x_2^2-t)=\frac{\beta^2}{8} \\ x_1x_2-t=0 \\ 2x_1^2-2x_2^2=\beta \end{cases}$$

Furthermore, $t>0$ because $t(x_1^2+x_2^2)=t^2+\frac{\beta^2}{8}$ and x_1 and x_2 have the same sign. Putting $\gamma=\frac{\beta}{2}$ and $\lambda=t+\frac{\beta^2}{8t}$, we find that $x_1^2=\frac{\lambda+\gamma}{2}$, $x_2^2=\frac{\lambda-\gamma}{2}$ and $t^2=\frac{\lambda^2-\gamma^2}{4}$. Hence, we get that $3t^4=\frac{\beta^4}{64}$. Thus $(\frac{\beta^2}{8},0,\beta,0,0)$ has two preimages $q_1=(t_0,a_1,b_1,0,0)$ and $q_2=(t_0,a_2,b_2,0,0)$, where $t_0>0$, $a_1,b_1>0$ and $a_2,b_2<0$. An easy computation shows that $DJ(q_i)=-128b_i^2t_0(a_i-b_i)^2$. Finally we find that $\deg_0 J=-2$. Corollary 3.17 gives that $\chi(W^{\varepsilon}_{(f-t,g-t)})=-2$.

Let us now compute $\chi(W^{\varepsilon}_{(f-t,g-\frac{1}{4}t)})$. The mappings H and J are :

$$\begin{aligned} H(t,x)=(x_1^2+x_2^2+x_3^2-x_4^2-t,x_1x_2+x_3x_4-\frac{1}{4}t,& \\ 2x_1^2-2x_2^2+2x_3^2+2x_4^2,-4x_2x_3,4x_2x_4),& \end{aligned}$$

$$J(t,x) = (t \cdot (x_1^2 + x_2^2 + x_3^2 - x_4^2 - t), x_1x_2 + x_3x_4 - \frac{1}{4}t,$$
$$2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4).$$

We use the same technics as in the previous example. We find that H and J have an isolated root at the origin. If $\beta < 0$ then $(0,0,\beta,0,0)$ has two preimages by H : $p_1 = (t_0, a_1, b_1, 0, 0)$ and $p_2 = (t_0, a_2, b_2, 0, 0)$ where $t_0 > 0$, $a_1, b_1 > 0$ and $a_2, b_2 < 0$. A computation gives that $DH(p_i) = -48b_i^2 t_0$, which implies that $\deg_0 H = -2$. Let us search the preimages of $(\frac{\beta^2}{8}, 0, \beta, 0, 0)$, $\beta < 0$, by J. If (t,x) is such a preimage then necessarly $x_2 \neq 0$, $x_3 = x_4 = 0$ and $t > 0$. Moreover x_1, x_2 and t satisfy the system :

$$\begin{cases} t(x_1^2 + x_2^2 - t) = \frac{\beta^2}{8} \\ x_1x_2 - \frac{1}{4}t = 0 \\ 2x_1^2 - 2x_2^2 = \beta \end{cases}$$

This gives that $-\frac{3}{4}t^2 = \frac{\beta^4}{63t^2}$, a contradiction. We have proved that $\deg_0 J = 0$. Applying Corollary 3.17, we obtain that $\chi(W^\varepsilon_{(f-t, g-\frac{1}{4}t)}) = -2$ if $t > 0$ and $\chi(W^\varepsilon_{(f-t, g-\frac{1}{4}t)}) = 2$ if $t < 0$.

References

[AFN1] AOKI, K., FUKUDA, T., NISHIMURA. T. : On the number of branches of the zero locus of a map germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^{n-1}, 0)$. *Topology and Computer Science: Proceedings of the Symposium held in honor of S. Kinoshita, H. Noguchi and T. Homma on the occasion of their sixtieth birhtdays* (1987), 347-363.

[AFN2] AOKI, K., FUKUDA, T., NISHIMURA. T. : An algebraic formula for the topological types of one parameter bifurcation diagrams, *Archive for Rational Mechanics and Analysis* **108** (1989), 247-265.

[AFS] AOKI, K., FUKUDA, T., SUN, W.Z. : On the number of branches of a plane curve germ, *Kodai Math. Journal* **9** (1986), 179-187.

[DLNS] DUDZINSKI, P., LECKI, A., NOWAK-PRZYGODZKI, P., SZAFRANIEC, Z. : On the topological invariance of the Milnor number mod 2, *Topology* **32** (1993), 573-576.

[Du1] DUTERTRE, N. : Degree formulas for a topological invariant of bifurcations of function germs, *Kodai Mathematical Journal* **23**, no. 3 (2000), 442-461.

[Du2] DUTERTRE, N. : On the Milnor fibre of a real map-germ, *Hokkaido Mathematical Journal* **31** (2002), 301-319.

[Du3] DUTERTRE, N. : Courbures et singularités réelles, *Commentarii Mathematici Helvetici* **77** (2002), 846-863.

[Du4] DUTERTRE, N. : On topological invariants associated with a polynomial with isolated critical points, *Glasg. Math. J.* **46**, no. 2 (2004), 323-334.

[Fu] FUKUI, T. : An algebraic formula for a topological invariant of bifurcation of 1-parameter family of function-germs, in *Stratifications, singularities, and differential equations, II (Marseille, 1990; Honolulu, HI, 1990)*, Travaux en cours **55**, Hermann, Paris, 45-54 1997.

[FK] FUKUI, T., KHOVANSKII, A. : Mapping degree and Euler characteristic, *Kodai Math. J.* **29**, no. 1 (2006), 144-162.

[Gr] GREUEL, G.M. : Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständingen Durschnitten, *Math. Annalen* **214** (1975), 235-266.

[Ha] HAMM, H. : Lokale topologische Eigenschaften komplexer Raume, *Math. Ann.* **191** (1971), 235-252.

[Kh] KHIMSHIASHVILI, G.M. : On the local degree of a smooth map, *Soobshch. Akad. Nauk Gruz. SSR* **85** (1977), 309-311.

[Le] LE DUNG TRANG : Calcul du nombre de Milnor d'une singularité isolée d'intersection complète, *Funct. Anal. Appl.* **8** (1974), 45-52.

[Mi] MILNOR, J. : Singular points of complex hypersurfaces, *Ann. Math. Stud.* **61**, Princeton University Press (1968).

[Sz1] SZAFRANIEC, Z. : On the number of branches of a 1-dimensional semi-analytic set, *Kodai Math. Journal* **11** (1988), 78-85.

[Sz2] SZAFRANIEC, Z. : A formula for the Euler characteristic of a real algebraic manifold, *Manuscripta Mathematica* **85** (1994), 345-360.

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